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The essential norm of a composition operator mapping into the Q_s-space ☆

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Abstract

An asymptotic formula, in terms of a global integral condition, for the essential norm $\|C_\varphi\|_e$ of the composition operator $C_\varphi(f) = f \circ \varphi$ mapping from the weighted Bergman space A^p_α, $1 < p \leqslant 2$, or the weighted Dirichlet space $\mathcal{D}_\alpha$ into the Möbius invariant Q_s-space is established. Moreover, it is shown that if C_φ is bounded from the classical Dirichlet space to *BMOA*, then

$$\|C_\varphi\|_e^2 = \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(z)\right|^2 N_{\varphi\circ\varphi_b}(z)\,dA(z) = \limsup_{|z|\to 1^-} \sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b}(z),$$

where $N_{\varphi\circ\varphi_b}(z)$ is the Nevanlinna counting function for $\varphi\circ\varphi_b$, and $\varphi_b(z) = (b-z)/(1-\bar{b}z)$.

Keywords: Q_s-space; Bergman space; Dirichlet space; Composition operator; Essential norm; Nevanlinna counting function

1. Introduction and results

Every analytic self-map φ of the open unit disc $\mathbb{D}$ in the complex plane induces the composition operator C_φ, defined by $C_\varphi(f) := f \circ \varphi$, acting on the space of all analytic functions in $\mathbb{D}$. It is well known that any such operator is a bounded linear operator on the classical Bergman and

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Hardy spaces. Probably the first papers strongly related to composition operators were [19,21]. For the theory of composition operators in spaces of analytic functions, see [5,23].

J.H. Shapiro [22] showed that the essential norm of C_φ, the distance of C_φ in the operator norm from compact operators, acting on the Hardy space H^2 equals to

$$\limsup_{|z|\to 1^-} \frac{N_\varphi(z)}{\log\frac{1}{|z|}}.$$

Writing $A^2_{-1} := H^2$, the space H^2 is identified with the limit space of the weighted Bergman space A^2_α as $\alpha \to -1^+$ [30]. With this in mind, P. Poggi-Corradini [20] generalized Shapiro's result for A^2_α when $\alpha \in \{-1, 0, 1\}$ by using contractive zero-divisors constructed by H. Hedenmalm [11,12]: the essential norm of C_φ acting on A^2_α equals to

$$\limsup_{|z|\to 1^-} \frac{N_{\varphi,2+\alpha}(z)}{\left(\log\frac{1}{|z|}\right)^{2+\alpha}}.$$

Here $N_{\varphi,\gamma}(z)$ denotes the generalized Nevanlinna counting function for the self-map φ, and $N_\varphi(z) := N_{\varphi,1}(z)$ is the Nevanlinna counting.

The purpose of this note is to study the essential norm of a composition operator mapping into the Möbius invariant Q_s-space. The approach taken here comes from Shapiro's work [22] on the essential norm of a composition operator acting on the Hardy space H^2. The main result is Theorem 1, which gives an asymptotic formula for the essential norm of the bounded operator C_φ acting from the weighted Bergman or Dirichlet space into Q_s-space in terms of a global integral condition. This yields a known characterization of compact composition operators. In the special case when the target space is $BMOA = Q_1$, the space of analytic functions of bounded mean oscillation, another asymptotic formula for the essential norm is given in Theorem 2. This one is similar to the formulas established by Shapiro and Poggi-Corradini. The proof of Theorem 2 yields Proposition 3 which gives characterizations of bounded and compact composition operators acting from the weighted Bergman or Dirichlet space into *BMOA*. Further, if C_φ acts from the classical Dirichlet space into *BMOA*, then the obtained formulas for the essential norm are not only asymptotic but they are exact by Corollary 4. It is also shown, as an example, that if the symbol φ is an inner function, then C_φ acting from the classical Besov space B^p, $1 < p \leqslant 2$, into *BMOA* is bounded, and the essential norm satisfies $\|C_\varphi\|_e \geqslant 2^{-1/2}$, so that C_φ is not compact.

Throughout this note functions denoted by f are always assumed to be analytic in $\mathbb{D}$, and φ denotes an analytic self-map of $\mathbb{D}$ so that $\varphi(\mathbb{D}) \subset \mathbb{D}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of those f for which

$$\|f\|_{A^p_\alpha} := \left((\alpha+1)\int_{\mathbb{D}} \left|f(z)\right|^p \left(1-|z|^2\right)^\alpha \, dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA is the element of the Lebesgue area measure normalized such that $A(\mathbb{D}) = 1$. For the theory of Bergman spaces, see [7,12]. A function f belongs to the Dirichlet type space $\mathcal{D}^p_\alpha$ if $f' \in A^p_\alpha$, and the true norm in $\mathcal{D}^p_\alpha$ is defined by $\|f\|_{\mathcal{D}^p_\alpha} := \|f'\|_{A^p_\alpha} + |f(0)|$ when $p \geqslant 1$. The classical Dirichlet space $\mathcal{D}$ is $\mathcal{D}^2_0$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is $\mathcal{D}^2_\alpha$, and the classical Besov space B^p is $\mathcal{D}^p_{p-2}$. Moreover, it is well known that $A^p_\alpha = \mathcal{D}^p_{p+\alpha}$ and

$$C^{-1}\|f\|_{A^p_\alpha} \leqslant \|f\|_{\mathcal{D}^p_{p+\alpha}} \leqslant C\|f\|_{A^p_\alpha} \tag{1.1}$$

for a positive constant C depending only on p and α. See, for example, [31].

Let the Green's function of $\mathbb{D}$, with a logarithmic singularity at $a \in \mathbb{D}$, be defined by $g(z,a) := -\log|\varphi_a(z)|$, where $\varphi_a(z) := (a-z)/(1-\bar{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the origin and the point a. The automorphism φ_a is its own inverse and satisfies the well-known identities

$$1 - \left|\varphi_a(z)\right|^2 = \left|\varphi_a'(z)\right|\left(1-|z|^2\right) = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

For $0 \leqslant s < \infty$, the Möbius invariant subspace Q_s of the weighted Dirichlet space $\mathcal{D}_s$ consists of those f for which

$$\|f\|_{Q_s} := \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 g^s(z,a)\,dA(z)\right)^{\frac{1}{2}} + \left|f(0)\right| < \infty.$$

The space Q_1 coincides with $BMOA$, the space of analytic functions of bounded mean oscillation, and if $s > 1$, then Q_s reduces to the Bloch space. The Q_s-spaces were introduced in [1,3], see [29] for more recent developments.

The following characterizations of bounded and compact composition operators acting from $\mathcal{D}^p_\alpha$ to Q_s have been essentially proved in [13,25,28]. Recall that a linear operator is compact if it takes a bounded set to relatively compact (precompact) set.

Theorem A. *Let $0 < p \leqslant 2$, $-1 < \alpha < \infty$ and $0 \leqslant s < \infty$. Then $C_\varphi : \mathcal{D}^p_\alpha \to Q_s$ is bounded if and only if*

$$\sup_{a,b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'\left(\varphi(z)\right)\right|^{\frac{2}{p}(\alpha+2)} \left|\varphi'(z)\right|^2 g^s(z,b)\,dA(z) < \infty,$$

and $C_\varphi : \mathcal{D}^p_\alpha \to Q_s$ is compact if and only if

$$\lim_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'\left(\varphi(z)\right)\right|^{\frac{2}{p}(\alpha+2)} \left|\varphi'(z)\right|^2 g^s(z,b)\,dA(z) = 0.$$

The essential norm $\|C_\varphi\|_e$ of a bounded operator C_φ is the distance in the operator norm of C_φ from compact operators, that is, $\|C_\varphi\|_e := \inf_K \|C_\varphi - K\|$, where the infimum is taken over all admissible compact operators K. Thus the essential norm of C_φ equals to zero if and only if C_φ is compact. For results on the essential norm of a composition operator acting on Bergman, Bloch and Hardy spaces, see, for instance, [4,6,10,14,17,18,20,22], and the references therein.

It is natural to expect that the essential norm $\|C_\varphi\|_e$ of a bounded composition operator C_φ acting from $\mathcal{D}^p_\alpha$ to Q_s is expressible in terms of a formula derived from the integral conditions in Theorem A. This is indeed the case as Theorem 1 below shows, and therefore the following notation is adopted through a change of variables (Lemma B below):

$$A(p,\alpha,s) := \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{\frac{2}{p}(\alpha+2)} N_{\varphi\circ\varphi_b,s}(w)\,dA(w).$$

Recall that the generalized Nevanlinna counting function for the self-map φ is the function

$$N_{\varphi,s}(w) := \sum_{z\in\varphi^{-1}\{w\}} \left(\log\frac{1}{|z|}\right)^s, \quad w \in \mathbb{D} \setminus \left\{\varphi(0)\right\},\ s > 0,$$

where $z \in \varphi^{-1}\{w\}$ is repeated according to the multiplicity of the zero of $\varphi - w$ at z. The Nevanlinna counting function is then $N_\varphi(z) := N_{\varphi,1}(z)$.

Theorem 1. *Let $1 < p \leqslant 2$, $-1 < \alpha < \infty$ and $0 < s < \infty$. If $C_\varphi : \mathcal{D}^p_\alpha \to Q_s$ is bounded, then there exists a positive constant c, depending only on p and α, such that*

$$A(p,\alpha,s) \leqslant \|C_\varphi\|_e^2 \leqslant cA(p,\alpha,s). \tag{1.2}$$

Taking into account the inequalities in (1.1), Theorem 1 implies that the essential norm of C_φ acting from the weighted Bergman space A^p_α, $1 < p \leqslant 2$, to Q_s satisfies the asymptotic equality

$$\|C_\varphi\|_e^2 \simeq \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{\frac{2}{p}(p+\alpha+2)} N_{\varphi\circ\varphi_b,s}(w)\, dA(w),$$

where the symbol $\simeq$ means that the quantities in other sides of the symbol are comparable, that is, their quotient is bounded and bounded away from zero. In the case when $C_\varphi : \mathcal{D}_\alpha \to Q_s$ is bounded, the inequalities in (1.2) yield $\|C_\varphi\|_e^2 \simeq A(2,\alpha,s)$.

The Sub-mean Value Property of the Nevanlinna counting function N_φ (Lemma C below) can be used to obtain another asymptotic formula, similar to the formulas established by Shapiro and Poggi-Corradini, for the essential norm of C_φ in the special case when the target space is *BMOA*. For this purpose, it is defined

$$B(p,\alpha,s) := \limsup_{|w|\to 1^-} \frac{\sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b,s}(w)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}}.$$

Theorem 2. *Let $1 < p \leqslant 2$ and $-1 < \alpha < \infty$ such that $p \leqslant 2 + \alpha$. If $C_\varphi : \mathcal{D}^p_\alpha \to BMOA$ is bounded, then*

$$\frac{B(p,\alpha,1)}{c} \leqslant A(p,\alpha,1) \leqslant \|C_\varphi\|_e^2 \leqslant \frac{B(p,\alpha,1)}{\alpha+1}, \tag{1.3}$$

where $c := c(p,\alpha) \geqslant 1$ depends only on p and α, and satisfies $c(p, p-2) = 1$.

If the target space is Q_s, $0 \leqslant s < 1$, then the proof of Theorem 2 shows that $\|C_\varphi\|_e^2 \leqslant (\alpha+1)^{-1}B(p,\alpha,s)$. Moreover, the assumption $p \leqslant 2+\alpha$ is only used to obtain the constant $(\alpha+1)^{-1}$ in the upper bound of the essential norm. If $p > 2+\alpha$, then the right most inequality in (1.3) should read $\|C_\varphi\|_e^2 \leqslant 2^{2-\frac{2}{p}(\alpha+2)} B(p,\alpha,1)$ which follows by using the inequality $1 - x^2 \leqslant -2\log x$, $0 < x \leqslant 1$, in an appropriate place in the proof.

The proofs of the upper bounds for the essential norm in Theorems 1 and 2 remain valid if $p = 1$, but it remains open whether the assertions on lower bounds hold in this case also.

It is also worth pointing out that the proof of Theorem 2, with appropriate modifications, and Theorem A yield the following characterization of bounded and compact composition operators acting from $\mathcal{D}^p_\alpha$ to *BMOA*.

Proposition 3. *Let $0 < p \leqslant 2$ and $-1 < \alpha < \infty$. Then $C_\varphi : \mathcal{D}^p_\alpha \to BMOA$ is bounded if and only if*

$$\sup_{w,b\in\mathbb{D}} \frac{N_{\varphi\circ\varphi_b}(w)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}} < \infty,$$

and $C_\varphi : \mathcal{D}^p_\alpha \to BMOA$ *is compact if and only if*

$$\lim_{|w|\to 1^-} \frac{\sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b}(w)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}} = 0.$$

If the symbol φ is an inner function, so that it satisfies $|\varphi(z)| = 1$ almost everywhere on the boundary of $\mathbb{D}$, then equality in Littlewood's inequality $N_\varphi(w) \leqslant -\log|\varphi_{\varphi(0)}(w)|$ holds for w outside a set of area measure zero by the classical 1935-theorem by Frostman, see [9] and also [15,22]. It follows that the essential norm of a bounded operator $C_\varphi : B^p \to BMOA$, $1 < p \leqslant 2$, satisfies

$$\|C_\varphi\|_e^2 = \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|(\varphi_a \circ \varphi_{\varphi(b)})'(z)\right|^2 \log\frac{1}{|z|}\, dA(z)$$

by Theorem 2. Choosing the sequence $\{b_n\} \subset \mathbb{D}$ such that $|\varphi(b_n)| \to 1^-$ as $n \to \infty$, and replacing a and b by $\varphi(b_n)$ and b_n, respectively, it follows that

$$\|C_\varphi\|_e^2 \geqslant \int_{\mathbb{D}} \log\frac{1}{|z|}\, dA(z) = \frac{1}{2}.$$

Moreover, since

$$\int_{\mathbb{D}} \left|\varphi_c'(z)\right|^2 \log\frac{1}{|z|}\, dA(z) \simeq \left(1-|c|^2\right)^2 \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{c}z|^4}\, dA(z) \simeq 1-|c|^2$$

for all $c \in \mathbb{D}$ by Forelli–Rudin estimates [12, Theorem 1.7], it follows that C_φ is bounded by Theorem A. Thus $C_\varphi : B^p \to BMOA$, $1 < p \leqslant 2$, is bounded but not compact if the symbol φ is an inner function.

If $C_\varphi : \mathcal{D} \to BMOA$ is bounded, then two different formulas for the essential norm are obtained by the inequalities in (1.3).

Corollary 4. *If* $C_\varphi : \mathcal{D} \to BMOA$ *is bounded, then*

$$\limsup_{|w|\to 1^-} \sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b}(w) = \|C_\varphi\|_e^2 = \limsup_{|a|\to 1^-} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^2 N_{\varphi\circ\varphi_b}(w)\, dA(w).$$

The remainder of this note is organized as follows. Section 2 contains the auxiliary results on Nevanlinna counting function and Carleson measures needed in the proofs of Theorems 1 and 2. The proofs themselves are presented in Sections 3 and 4.

2. Auxiliary results

The following change of variables formula plays an important role in the proofs. In this generality, it is a special case of the Area Formula [5, Theorem 2.32], see also [8,22,26].

Lemma B. *Let* g *be a positive measurable functions on* $\mathbb{D}$*, and let* φ *be an analytic self-map of* $\mathbb{D}$*. Then*

$$\int_{\mathbb{D}} (g\circ\varphi)(z)\left|\varphi'(z)\right|^2 \left(\log\frac{1}{|z|}\right)^s dA(z) = \int_{\mathbb{D}} g(w) N_{\varphi,s}(w)\, dA(w)$$

for all $0 < s < \infty$.

Another result needed is the Sub-mean Value Property of the Nevanlinna counting function [8,22].

Lemma C. *Let φ be an analytic self-map of $\mathbb{D}$, $\varphi(0) \neq 0$ and $0 < r < |\varphi(0)|$. Then*

$$N_\varphi(0) \leqslant \frac{1}{\pi r^2} \int\limits_{\Delta(0,r)} N_\varphi(z)\, dA(z).$$

Here and now on $\Delta(0,r)$ denotes the Euclidean disc centered at the origin and of radius $r \in (0,1)$.

A positive measure μ on $\mathbb{D}$ is a bounded t-Carleson measure, if

$$\sup_I \frac{\mu(S(I))}{|I|^t} < \infty, \quad 0 < t < \infty, \tag{2.1}$$

where $|I|$ denotes the arc length of a subarc I of the boundary of $\mathbb{D}$,

$$S(I) = \left\{ z \in \mathbb{D}\colon \frac{z}{|z|} \in I,\ 1 - |I| \leqslant |z| \right\}$$

is the Carleson box based on I, and the supremum is taken over all subarcs I such that $|I| \leqslant 1$.

The last auxiliary result needed is D. Luecking's [16] characterization of Carleson measures in terms of functions in the weighted Bergman spaces.

Theorem D. *Let μ be a positive measure on $\mathbb{D}$, and let $0 < p \leqslant q < \infty$. Then μ is a bounded $\frac{q}{p}(2+\alpha)$-Carleson measure if and only if there is a positive constant C, depending only on p and q, such that*

$$\int\limits_{\mathbb{D}} \left|f(z)\right|^q d\mu(z) \leqslant C \|f\|^q_{A^p_\alpha} \tag{2.2}$$

for all analytic functions f in $\mathbb{D}$, in particular for all $f \in A^p_\alpha$. Moreover, if μ is a bounded $\frac{q}{p}(2+\alpha)$-Carleson measure, then $C = C_1 C_2$, where $C_1 > 0$ depends only on p, q and α, and

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^{\frac{q}{p}(2+\alpha)}}. \tag{2.3}$$

It is known that

$$\sup_I \frac{\mu(S(I))}{|I|^t} \simeq \sup_{a \in \mathbb{D}} \int\limits_{\mathbb{D}} \left|\varphi_a'(z)\right|^t d\mu(z)$$

for all $0 < t < \infty$ [2,9], and if $t > 1$, then

$$\sup_I \frac{\mu(S(I))}{|I|^t} \simeq \sup_{a \in \mathbb{D}} \frac{\mu(D(a,r))}{(1-|a|^2)^t}$$

for any fixed $r \in (0,1)$, where $D(a,r)$ is the pseudohyperbolic disc $\{z \in \mathbb{D}\colon |\varphi_a(z)| < r\}$ with pseudohyperbolic center a and radius r [16]. The pseudo-hyperbolic disc $D(a,r)$ is an Euclidean disc centered at $(1-r^2)a/(1-|a|^2r^2)$ and of radius $(1-|a|^2)r/(1-|a|^2r^2)$ [9], in particular, $D(0,r) = \Delta(0,r)$.

3. Proof of Theorem 1

The lower bound in (1.2)

For $a \in \mathbb{D}$, define

$$f_a(z) := \int_0^z \left(\frac{1-|a|^2}{(1-\bar{a}w)^2} \right)^{\frac{\alpha+2}{p}} dw.$$

Then $f_a(0) = 0$, $\|f_a\|_{\mathcal{D}_\alpha^p} = \|(-\varphi_a')^{\frac{\alpha+2}{p}}\|_{A_\alpha^p} = 1$ by [27], and $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Moreover, since the space $\mathcal{D}_\alpha^p$ is reflexive when $1 < p < \infty$, the argument in [24, p. 318] may be used to show that $f_a \to 0$ weakly in $\mathcal{D}_\alpha^p$ as $|a| \to 1^-$. Therefore, if $K : \mathcal{D}_\alpha^p \to Q_s$ is compact, then

$$\begin{aligned} \|C_\varphi - K\| &\geqslant \limsup_{|a|\to 1^-} \left\| C_\varphi(f_a) - K(f_a) \right\|_{Q_s}^2 \\ &\geqslant \limsup_{|a|\to 1^-} \left\| C_\varphi(f_a) \right\|_{Q_s}^2 - \limsup_{|a|\to 1^-} \left\| K(f_a) \right\|_{Q_s}^2 \\ &= \limsup_{|a|\to 1^-} \left\| C_\varphi(f_a) \right\|_{Q_s}^2, \end{aligned}$$

and since $\lim_{|a|\to 1^-} |f_a(\varphi(0))| = 0$, Lemma B yields $A(p,\alpha,s) \leqslant \|C_\varphi\|_e^2$, that is, the lower bound in (1.2) for the essential norm is proved.

The upper bound in (1.2)

For an analytic function $f(z) = \sum_{k=0}^\infty a_k z^k$ in $\mathbb{D}$, define

$$T_n f(z) := \sum_{k=0}^n a_k z^k, \qquad R_n f(z) := \sum_{k=n+1}^\infty a_k z^k.$$

Since T_n is compact on $\mathcal{D}_\alpha^p$,

$$\|C_\varphi\|_e = \left\| C_\varphi(T_n + R_n) \right\|_e \leqslant \|C_\varphi T_n\|_e + \|C_\varphi R_n\|_e = \|C_\varphi R_n\|_e \leqslant \|C_\varphi R_n\|,$$

and it follows that

$$\|C_\varphi\|_e \leqslant \liminf_{n\to\infty} \|C_\varphi R_n\|.$$

Therefore, by Lemma B,

$$\begin{aligned} \|C_\varphi\|_e^2 &\leqslant \liminf_{n\to\infty} \|C_\varphi R_n\|^2 \leqslant \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leqslant 1} \left\| (C_\varphi R_n)(f) \right\|_{Q_s}^2 \\ &= \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leqslant 1} \left(\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left| (R_n f \circ \varphi)'(z) \right|^2 g^s(z,b)\, dA(z) + \left| R_n(f\circ\varphi)(0) \right| \right) \\ &= \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leqslant 1} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left| R_{n-1} f'\big(\varphi(z)\big) \right|^2 \left| \varphi'(z) \right|^2 g^s(z,b)\, dA(z) \\ &= \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p}\leqslant 1} \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left| R_{n-1} f'(w) \right|^2 N_{\varphi\circ\varphi_b,s}(w)\, dA(w), \end{aligned} \tag{3.1}$$

and since

$$\liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^p_\alpha}\leqslant 1} \sup_{b\in\mathbb{D}} \int\limits_{\Delta(0,r)} \left|R_{n-1}f'(w)\right|^2 N_{\varphi\circ\varphi_b,s}(w)\,dA(w)=0$$

for any $r\in(0,1)$, Theorem D yields

$$\|C_\varphi\|_e^2 \leqslant C_1 \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^p_\alpha}\leqslant 1} \|R_{n-1}f'\|^2_{A^p_\alpha} \sup_{b\in\mathbb{D}} C(b) \leqslant C_1 \sup_{b\in\mathbb{D}} C(b), \tag{3.2}$$

where C_1 is a positive constant and

$$C(b) := \sup_I \frac{1}{|I|^{\frac{2}{p}(2+\alpha)}} \int\limits_{S(I)\setminus\Delta(0,r)} N_{\varphi\circ\varphi_b,s}(w)\,dA(w).$$

Denote $d\mu(w) := N_{\varphi\circ\varphi_b,s}(w)\,dA(w)$ and $t := \frac{2}{p}(2+\alpha)$. It will be shown next that

$$C(b) = \sup_I \frac{\mu(S(I)\setminus\Delta(0,r))}{|I|^t} \leqslant 2 \sup_{|I|\leqslant 1-r} \frac{\mu(S(I))}{|I|^t}, \tag{3.3}$$

where $S(I)$ is the Carleson box defined after formula (2.1). To do this, consider the subarc $I := \{e^{i\phi}\colon\ \theta\leqslant\phi\leqslant\theta+|I|\}$ of the boundary of $\mathbb{D}$ such that $1-r<|I|\leqslant 1$, since otherwise $S(I)\setminus\Delta(0,r)=S(I)$. Let $n:=\max\{k\in\mathbb{N}\colon\ k(1-r)<|I|\}$, and define $I_j := \{e^{i\phi}\colon\ \theta+j(1-r)\leqslant\phi\leqslant\theta+(j+1)(1-r)\}$ for $j=0,\dots,n$. Then $|I_j|=1-r$ for $j=0,\dots,n$, and therefore

$$\begin{aligned}
\frac{\mu(S(I)\setminus\Delta(0,t))}{|I|^t} &= \frac{1}{|I|^t}\int\limits_{S(I)} \chi_{\mathbb{D}\setminus\Delta(0,r)}(w)\,d\mu(w) \\
&\leqslant \frac{1}{|I|^t}\left(\sum_{j=0}^{n-1}\int\limits_{S(I_j)} d\mu(w) + \int\limits_{S(I_n)} \chi_{\mathbb{D}\setminus\Delta(0,r)}(w)\,d\mu(w)\right) \\
&\leqslant \frac{1}{n^t}\left(\sum_{j=0}^{n}\frac{1}{|I_j|^t}\int\limits_{S(I_j)} d\mu(w)\right) \\
&\leqslant \frac{n+1}{n^t}\sup_{|I|\leqslant 1-r}\frac{\mu(S(I))}{|I|^t} \leqslant 2\sup_{|I|\leqslant 1-r}\frac{\mu(S(I))}{|I|^t},
\end{aligned}$$

from which (3.3) follows. Let now $w\in S(I)$, where $I=\{e^{i\phi}\colon\ \theta\leqslant\phi\leqslant\theta+|I|\}$, and choose $a:=(1-|I|)e^{i(\theta+\frac{|I|}{2})}$ so that $e^{i(\theta+\frac{|I|}{2})}$ is the midpoint of I. Since $|1-\bar{a}w|\leqslant 1-|a|^2+|w-a|$, and

$$\begin{aligned}
|w-a|^2 &\leqslant \left|e^{i(\theta+|I|)} - \big(1-|I|\big)e^{i(\theta+\frac{|I|}{2})}\right|^2 \\
&= 1-2\big(1-|I|\big)\Re\big(e^{-i\frac{|I|}{2}}\big) + \big(1-|I|\big)^2 \\
&\leqslant 1-2\big(1-|I|\big)\left(1-\frac{|I|^2}{8}\right) + \big(1-|I|\big)^2 \leqslant \frac{5}{4}|I|^2,
\end{aligned}$$

one has $|\varphi_a'(w)|\geqslant(10|I|)^{-1}$ for all $w\in S(I)$, and it follows that

$$\sup_{|I|\leqslant 1-r}\frac{\mu(S(I))}{|I|^t} \leqslant 10^t \sup_{|a|\geqslant r}\int\limits_{\mathbb{D}} \left|\varphi_a'(w)\right|^t d\mu(w). \tag{3.4}$$

Finally, (3.2), (3.3) and (3.4) yield

$$\|C_\varphi\|_e^2 \leqslant 2 \cdot 10^{\frac{2}{p}(2+\alpha)} C_1 \sup_{b\in\mathbb{D}} \sup_{|a|\geqslant r} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{\frac{2}{p}(2+\alpha)} N_{\varphi\circ\varphi_b,s}(w)\, dA(w),$$

and the upper bound in (1.3) for the essential norm follows by letting $r \to 1^-$.

4. Proof of Theorem 2

The second inequality in (1.3) follows by Theorem 1, so it suffices to show the first inequality and the upper bound for the essential norm in (1.3).

The first inequality in (1.3)

An application of Lemma C for the Nevanlinna counting function for the self-map $\varphi_w \circ \varphi \circ \varphi_b$ yields

$$\begin{aligned}\limsup_{|w|\to 1^-} \frac{\sup_{b\in\mathbb{D}} N_{\varphi\circ\varphi_b}(w)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}} &\leqslant \frac{1}{r^2} \limsup_{|w|\to 1^-} \frac{\sup_{b\in\mathbb{D}} \int_{\Delta(0,r)} N_{\varphi_w\circ\varphi\circ\varphi_b}(u)\, dA(u)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}} \\ &= \frac{1}{r^2} \limsup_{|w|\to 1^-} \frac{\sup_{b\in\mathbb{D}} \int_{D(w,r)} |\varphi_w'(z)|^2 N_{\varphi\circ\varphi_b}(z)\, dA(z)}{\left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2}} \end{aligned} \tag{4.1}$$

for all $0 < r < 1$. Fix $r = \frac{1}{2}$. Since $\frac{1}{2}(1-|w|^2) \leqslant -\log|w| \leqslant \frac{1}{t}(1-|w|^2)$ for $0 < t \leqslant |w| \leqslant 1$, and $\frac{1}{4}(1-|w|^2)^{-1} \leqslant |\varphi_w'(z)| \leqslant \frac{9}{4}(1-|w|^2)^{-1}$ for all $z \in D(w, \frac{1}{2})$, the first inequality in (1.3) follows. If $\alpha = p-2$ then the power $\frac{2}{p}(\alpha+2)-2$ in each denominator of the quotients in (4.1) equals to zero, and therefore the first inequality in (1.3) with $c(p, p-2) = 1$ follows as $r \to 1^-$.

The upper bound in (1.3)

It will be shown next that, if $C_\varphi : \mathcal{D}_\alpha^p \to Q_s$ is bounded, then $\|C_\varphi\|_e^2 \leqslant (\alpha+1)^{-1} B(p,\alpha,s)$. The upper bound in (1.3) then follows by choosing $s = 1$. If $B(p,\alpha,s) = 0$, then there is nothing to prove by Theorem A since

$$\begin{aligned} A(p,\alpha,s) &\leqslant B(p,\alpha,s) \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|\varphi_a'(w)\right|^{\frac{2}{p}(\alpha+2)} \left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2} dA(w) \\ &\simeq B(p,\alpha,s) \sup_{a\in\mathbb{D}} \left(1-|a|^2\right)^{\frac{2}{p}(\alpha+2)} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\frac{2}{p}(\alpha+2)-2}}{|1-\bar{a}w|^{\frac{4}{p}(\alpha+2)}}\, dA(w) \\ &\simeq B(p,\alpha,s) \end{aligned}$$

by Forelli–Rudin estimates [12, Theorem 1.7]. Assume now that $B(p,\alpha,s) > 0$. For a given $\varepsilon > 0$ there is $r_\varepsilon \in (0,1)$ such that

$$\frac{N_{\varphi\circ\varphi_b,s}(z)}{\left(\log\frac{1}{|z|}\right)^{\frac{2}{p}(\alpha+2)-2}} \leqslant (1+\varepsilon) B(p,\alpha,s) \tag{4.2}$$

for all $|z| \geqslant r_\varepsilon$ and $b \in \mathbb{D}$. The reasoning in (3.1) together with the inequality (4.2) yields

$$\begin{aligned}
&\|C_\varphi\|_e^2 \\
&\quad \leqslant (1+\varepsilon)B(p,\alpha,s) \liminf_{n\to\infty} \sup_{\|f\|_{\mathcal{D}^p_\alpha}\leqslant 1} \int\limits_{\mathbb{D}\setminus\Delta(0,r_\varepsilon)} \left|R_{n-1}f'(w)\right|^2 \left(\log\frac{1}{|w|}\right)^{\frac{2}{p}(\alpha+2)-2} dA(w) \\
&\quad \leqslant \frac{1+\varepsilon}{r_\varepsilon^{\frac{2}{p}(\alpha+2)-2}} B(p,\alpha,s) \sup_{\|f\|_{\mathcal{D}^p_\alpha}\leqslant 1} \int\limits_{\mathbb{D}} \left|f'(w)\right|^2 \left(1-|w|^2\right)^{\frac{2}{p}(\alpha+2)-2} dA(w),
\end{aligned}$$

where the last inequality follows by the inequality $\log\frac{1}{|w|} \leqslant \frac{1}{r_\varepsilon}(1-|w|^2)$ for $0 < r_\varepsilon \leqslant |w| \leqslant 1$. But now $|f'(w)|(1-|w|^2)^{\frac{\alpha+2}{p}} \leqslant \|f'\|_{A^p_\alpha}$ by [27, Corollary], and therefore

$$\|C_\varphi\|_e^2 \leqslant \frac{1+\varepsilon}{r_\varepsilon^{\frac{2}{p}(\alpha+2)-2}} B(p,\alpha,s) \sup_{\|f\|_{\mathcal{D}^p_\alpha}\leqslant 1} \|f'\|_{A^p_\alpha}^{2-p} \int\limits_{\mathbb{D}} \left|f'(w)\right|^p \left(1-|w|^2\right)^\alpha dA(w).$$

Since $\varepsilon \to 0$ as $r_\varepsilon \to 1^-$, it follows that $\|C_\varphi\|_e^2 \leqslant (\alpha+1)^{-1}B(p,\alpha,s)$.

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